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Artículo Original

Fractional Model to estimate the turbulence index in pipelines

Modelo fraccional para estimación del índice de turbulencia en oleoductos

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Abstract

Turbulence is a phenomenon whose description is still an open question, despite presenting itself in nature and many industrial processes. While in the laminar regime, the flow behavior is described successfully through the phenomenological equations of conservation of the moment. In a turbulent regime, this approach fails since the fluctuations around the average speed are magnified and cannot neglect. Pressure drop determinations are essential in pipelines since they are related to an oil country's production and economy. This paper proposes a model to estimate the turbulence speed profile for flows in tubes as a function of the Reynolds number. The moment's conservation equations are fractional differential equations, where the hypothesis established that the order of the derivative depends on the intensity of the fluctuations. The theoretical results by our Model correspond qualitatively with those expected, allowing hypothesis acceptance.

Keywords: turbulent regime, laminar sublayer, fractional derivatives, Reynolds number, pipeline flow

Resumen

La turbulencia es un fenómeno cuya descripción sigue siendo una cuestión abierta, a pesar de presentarse en la naturaleza y en muchos procesos industriales. Mientras se encuentra en el régimen laminar, el comportamiento del flujo se describe con éxito a través de las ecuaciones fenomenológicas de conservación del momento. En un régimen turbulento, este enfoque falla ya que las fluctuaciones alrededor de la velocidad promedio se magnifican y no se pueden ignorar. Las determinaciones de caída de presión son importantes en oleoductos ya que se relaciona con la producción y economía de un país petrolero. El presente trabajo propone un modelo para estimar el perfil de velocidad de turbulencia para flujos en tubos en función del número de Reynolds. Las ecuaciones de conservación del momento son ecuaciones diferenciales fraccionarias, donde la hipótesis establece que el orden de la derivada depende de la intensidad de las fluctuaciones. Los resultados teóricos predichos por el modelo propuesto se corresponden cualitativamente con los esperados, lo que permitió aceptar la hipótesis.

Palabras clave: Régimen turbulento, subcapa laminar, derivadas fraccionales, número de Reynolds, flujo en oleoducto

Glossary of Terms

ρ Density (kg/m^3)
V Volume
μ Viscosity (*Pa.s*)
v_i velocity of the flow in *i-axis* (*m/s*)
θ angle of the velocity vector
R_e Reynold number
p Pressure (*Pa*)
r Radii at the point analised (*m*)
h Height difference (*m*)
g Gravity force
V Average fluid velocity (*m/s*)
R Total radii of the pipeline (*m*)
δ thickness of the laminar sublayer (*m*)
Φ Pressure variation related to pressure
k Parameter obtained by practical criteria or experimental observation
V^i Average velocity concerning to the parameter *i*.
γ Dimensionless parameter related to $\frac{r}{R}$
Σ Dimensionless parameter related to $\frac{z}{L}$
φ Dimensionless parameter related to $\frac{v_z}{\langle v_z \rangle}$
D_x^n Operator
$\Gamma(\alpha)$ Gamma function
f Friction factor
$\alpha(R_e)$ Order of the fractional derivative for R_e
ΔP Pressure drop (*Pa*)
I Turbulence index
b Interface position (*m*)

1. Introduction

The speed of a fluid that is transported through a tube depends on the spatial position, being practically equal to zero in the tube wall's vicinity and having a maximum value in its center. At low speeds, the temporal fluctuations of the local velocity are negligible, and the flow layers move without any mixing between them, which is known as the laminar regime. The magnitude of the fluctuations increases with the average velocity until reaching a limit value. The blending between the flow layers and the formation of flow patterns, such as vortices and eddies, occurs, known as the regime turbulent [1-3], as is shown in Figure 1.

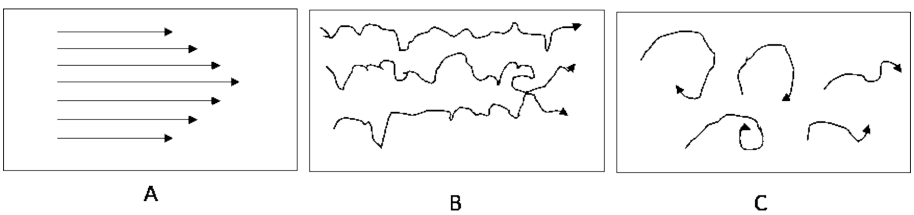


Fig.1 Speed profile for the laminar regime (A), a transition region between laminar and turbulent regime (B), and developed turbulent regime flow (C)

Experimentally, it has been found that the bifurcation point between the laminar regime and the turbulent regime depends on the Reynolds number [1], a non-dimensional parameter that quantifies the relationship between inertial forces and viscous forces, and that for the case of flow in tubes. It is given by:

$$R_e = \frac{2\rho VR}{\mu} \qquad (1)$$

The regime laminar for Re < 2100, 2100 < Re < 4000 is the transition regime, and Re > 4000 corresponds to the turbulent flow regime.

Navier-Stokes equations are simplified, obtaining the differential equation for this case [1] :

$$-\frac{\partial p}{\partial z} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \rho ghcos(\theta) = 0 \qquad (2)$$

Where p [Pa] is the pressure, v [m/s] is the velocity of the flow, r [m] and z [m] are the radii and the tube length, g is the acceleration of gravity, h [m] the height difference, θ the angle of the velocity vector concerning the direction of gravity and μ [Pa·s] and ρ [kg/m^3] are viscosity and density of the fluid, respectively. Equation (1) correctly describes the laminar flow regime, where a curved profile is obtained that corresponds to the observed results given by V (the average fluid velocity):

$$v = V\left(1 - \frac{r^2}{R^2}\right) \qquad (3)$$

Equation (3) is inappropriate for the turbulent regime. It must consider fluctuations and the velocity profile shows a flattening in the regions near the center of the tube [2,3], as is shown in Figure 2.

The understanding and mathematical description of the turbulence phenomenon remains an open question due to two fundamental factors [4]. The first is the need to explicitly consider the fluctuations of velocity in all the vector components, which implies that stochastic formalism in a deterministic description is already quite complicated. The second factor is unknown; this is the relationship between the fluctuations magnitude and the average speed obtained from integrating the velocity profile, a solution for the area flow. The models that have been proposed to describe the turbulent speed profile attempt to simplify these complications and involve in one way or another the inclusion of empirical parameters or relationships determined from experimental observations. One of these models is based on separating the profile two regions, close to the tube wall and known as the laminar sublayer. The second is in the vicinity of the fluid center, which is known as a turbulent sublayer. According to this Model, the speed profile is given by [1]:

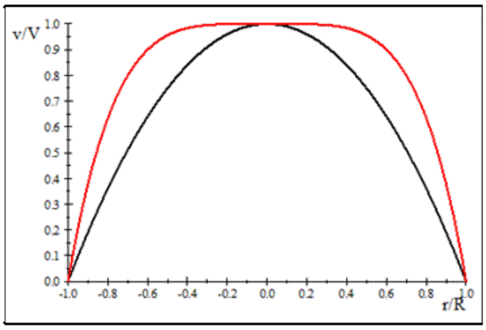


Fig.2 Speed profile for laminar speed (-) and turbulent rate (-), where V [m/s] is the average fluid velocity and R is the radius of the tube [m]

$$V^t = \left(\frac{R\Phi}{2\rho k^2}\right)^{0.5} ln\frac{(R-r)}{(R-\delta)} + \frac{1}{4}\frac{\Phi}{\mu}(R^2 - \delta^2) \quad : \quad r \leq \delta \qquad (4)$$

$$V^{\mu} = \frac{1}{4}\frac{\Phi R^2}{\mu}\left(1-\left(\frac{r}{R}\right)^2\right) \quad : \quad \delta \le r \tag{5}$$

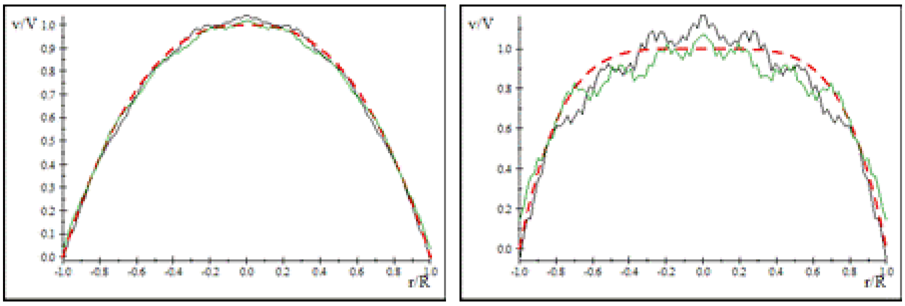
where $\Phi = -\frac{dp}{dz}$, k is a parameter of the Model, and δ [m] is the thickness of the laminar sublayer.

This Model's fundamental limitation is that the transition region between the laminar and turbulent sublayer is diffuse. The values of k and δ are estimated by practical criteria or from experimental observations. In an attempt to overcome these limitations, Computational Fluid Dynamics and complexity theory tools have been used to describe the turbulent speed profile [5-14]; in this paper, analytical models which can be used practically in the engineering field are proposed.

The research reported in this paper establishes the hypothesis that the order of the derivative involved in the conservation equations of momentum for flow in steady-state tubes is related to the regime flow. Equation whose solution allows estimating the average time profile of turbulent speed are written as fractional differentials. This research aimed to obtain a model for the velocity profile in turbulent flow to flow in tubes, establish a relationship between the derivative and the Reynolds number, and get a correlation to estimate the thickness of the laminar sublayer. The aim proposed in this work is to visualize the average velocity profile as a deterministic function that depends on a fractal space's coordinates, whose dimension depends on the Reynolds number. In this way, solving a stochastic partial differential equation, a fractional differential equation is obtained where the order is related to the fractal dimension, estimated from getting the average velocity and of the experimental observations related to the behavior of this velocity with the directing force applied in the system [7, 12, 13].

2. Considerations established for obtaining the Model

The velocity profile for turbulent flow shows a random behavior caused by fluctuations in each velocity component, which varies temporarily even if the system is in a stationary state. In this sense, while the actual value of the speed in a position changes temporarily, the average local rate remains invariant over time. Therefore, it is possible to define a temporal average velocity profile, as illustrated in Figure 3, whose distortion concerning the laminar profile increases with the Reynolds number and the magnitude of the fluctuations.



A B

Fig.3 Instantaneous velocity profile (continuous line) for two times values and average temporal velocity profile (dashed line), where 2100 > Re (A) and Re > 2100 (B)

Each velocity profile is obtained for different time values and visualized as the realization of a stochastic process. The average magnitude of the fluctuations is constant over time, where the average velocity profile is obtained by averaging this process's accomplishments. This behavior can be described through stochastic formalism [7], which implies the solution of a stochastic partial differential equation where, in addition to the mathematical complication involved to find an exact analytical solution, knowledge of the relationship between the magnitude of the fluctuations and the average velocity in the tube, which is the dependent variable whose behavior is to be determined.

Fractional Equation solution

Obtaining the average velocity profile requires expressing the partial differential equation (3) based on the non-dimensional variables:

$$\gamma = \frac{r}{R} \quad (6)$$
$$\Sigma = \frac{z}{L} \quad (7)$$
$$\varphi = \frac{v_z}{\langle v_z \rangle} \quad (8)$$

and replace the first derivative with the operator representing the fractional derivative:

$$\mathrm{D}_x^n = \frac{\partial^n}{\partial x^n} \quad (9)$$

where n is a fractional number equal to or greater than 1, such that:

$$\Phi + \frac{1}{\gamma}\mathrm{D}_\gamma^\alpha\left(\gamma \mathrm{D}_\gamma^\alpha \varphi\right) = 0 \quad (10)$$

where:

$$V = \langle v_z \rangle = \frac{\int_0^1 \int_0^1 v_z(\gamma)\gamma d\gamma d\Theta}{\int_0^1 \int_0^1 \gamma d\gamma d\Theta} \quad (11)$$
$$\Phi = -\frac{1}{L}\frac{\partial p}{\partial \Sigma}\frac{1}{\mu}\frac{R^2}{\langle v_z \rangle} \quad (12)$$

Equation (9) can be expressed as a system of fractional differential equations:

$$\Phi\gamma + \mathrm{D}_\gamma^\alpha T = 0 \quad (13)$$
$$T = \gamma \mathrm{D}_\gamma^\alpha \varphi \quad (14)$$

with the boundary conditions $T(0) = 0$ (14) and $\varphi(1) = 0$ (15) whose exact analytical solution is given by :

$$T = -\frac{\Phi}{\alpha(\alpha+1)\Gamma(\alpha)}\gamma^{1+\alpha} \quad (15)$$
$$\varphi = \frac{\Phi}{2\alpha(1+\alpha)\Gamma(2\alpha)}\left(1-\gamma^{2\alpha}\right) \quad (16)$$

Where Γ (α) is the gamma function defined as:

$$\Gamma(\alpha) = \int_0^\infty \exp(-x)x^{\alpha-1}dx \quad (17)$$

The average flow rate is calculated by integrating the velocity profile given by equation (17). From considering the definition of the non-dimensional velocity given by equation (7), it is obtained:

$$1 = \frac{1}{2}\frac{\Phi}{\left(1+\alpha_\gamma\right)^2\Gamma(2\alpha_\gamma)} \quad (18)$$

Substituting non-dimensional parameter and solving that allows to obtain behavior as pressure drop as a function of fluid properties, the radius of the tube, the average velocity, and order of the derivative fractional:

$$\frac{\Delta p}{L} = 2\mu\frac{V}{R^2}(1+\alpha)^2\Gamma(2\alpha) \tag{19}$$

From a practical point of view, the ratio used to estimate friction pressure losses is :

$$\frac{\Delta P}{L} = 2fV^2\frac{\rho}{D} \tag{20}$$

where f is a non-dimensional parameter, called friction factor, which depends on the turbulence's intensity. The relationship between f and R_e was obtained from the experimental observations, and different statistical models have been proposed to describe expressing this correlation. One of the simplest models is:

$$f = \frac{64}{R_e} \quad : \quad (0 < R_e < 2.1\times10^3) \tag{21}$$

$$f = \frac{0.0791}{R_e^{\frac{1}{4}}} \quad : \quad (2.1\times10^3 < R_e < 10^5) \tag{22}$$

With (20) and (21) considering correlations (22) and (23), the behavior of the order of the fractional derivative with Reynolds number is:

$$\alpha(R_e) = \begin{cases} 0.35392(\ln R_e) - 5.3388\times10^{-3}(\ln R_e)^2 - 1.1525 & if \quad R_e \geq 2100 \\ 1 & if \quad 2100 \geq R_e \end{cases} \tag{23}$$

The speed profile for turbulent speed expressed as the average speed flow is:

$$\frac{v(r)}{V} = \frac{\alpha+1}{\alpha}\left(1-\left(\frac{r}{R}\right)^{2\alpha}\right) \tag{24}$$

3. Results and Discussion

Figure 4 shows the ratio behavior between the average speed and the maximum speed for the turbulence index, which is given by:

$$\frac{V}{V_{max}} = \frac{\alpha}{1+\alpha} \tag{25}$$

As can be seen, the value of this $\frac{V}{V_{max}}$, which is equal to 0.5 in the laminar regime, tends to 1 as the turbulence index increases. This result corresponds to the generally used practical criterion based on considering an utterly flat profile of the fluid velocity for a fully developed turbulent regime.

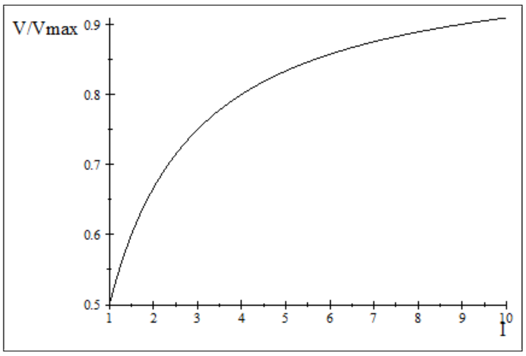


Fig.4 The behavior of the quotient between the average speed and maximum speed concerning the turbulence index I=α

Figure 5 shows the speed profiles for different Reynolds number values. It can be seeing the distortion of this profile concerning the corresponding laminar regime and how it increases with the Reynolds number. These results correspond qualitatively with the behaviors observed experimentally, so the results predicted by the proposed Model correspond to those expected.

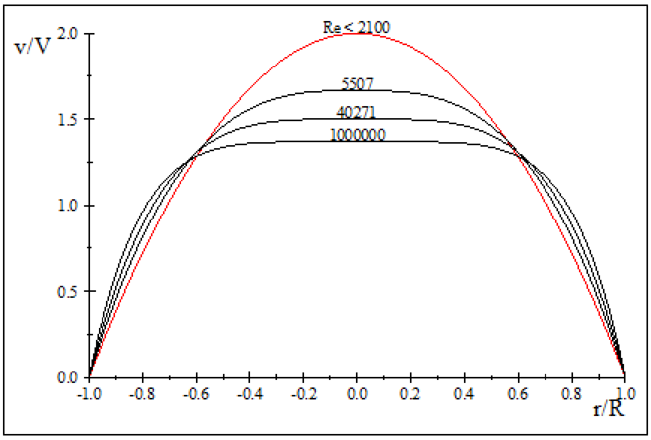


Fig.5 Speed profiles predicted by the Model considered as the Reynolds number parameter

Knowledge of the turbulent speed profile is of great practical importance to improve and increase industrial equipment's operating efficiency in which heat and mass transport processes occur. These processes occur at a higher speed in a turbulent regime since they favor convective transport mechanism type. In this way, the regions near the walls, where the fluid velocity is lower, are identified with the laminar boundary sublayer. In this area, transport processes slow down because the molecular transport mechanism predominates [1,11].

From (23) and (24) it is possible to identify the position b/R of the interface between the laminar and turbulent sublayer; the predicted velocity for laminar flow is equal to the expected speed for turbulent flow for the exact condition of the average rate, of such so that the position of the interface can be obtained from the relationship:

$$\frac{\alpha+1}{\alpha}\left(1-\left(\frac{b}{R}\right)^{2\alpha}\right)=2\left(1-\left(\frac{b}{R}\right)^{2}\right) \qquad (26)$$

Equation (26) has no exact solution, being necessary to apply numerical methods of solution.
Considering that the thickness of the laminar sublayer is given by:

$$\frac{\Delta}{R}=1-\frac{b}{R} \qquad (27)$$

applying statistical techniques and considering the relationship between the Reynolds number and the turbulence index, the following expression is proposed to estimate the thickness of the laminar sublayer:

$$\frac{\delta}{R} = 5.5 \times 10^{-4}(\ln R_e)^2 - 2 \times 10^{-2}\ln(R_e) + 0.6 \quad (28)$$

This equation allows predicting the value of the laminar boundary layer's width with the Reynolds number, where it decreases with the increase in turbulence corresponding to the expected value. The proposed Model has the following limitations: it is only valid in steady-state; it does not describe the behavior of the average magnitude of the fluctuations; it is helpful for flows in pipelines and, the Model obtained to predict the thickness of the laminar boundary sublayer is based on the spatial position at which the velocity for the laminar regime and the turbulent are equal. The fluctuations magnitude is implicitly included in the Reynolds number and inside the order of the fractional derivative. In further works, the theoretical results in Mexican pipelines pressure drop will be applied, where chemical products are dossed to enhance production, and transitional regimes are presented.

4. Conclusions

Considering that the partial differential equation that describes the flow in tubes is expressed in a fractional differential equation, a model was obtained to predict the average time profile in a turbulent flow. It was assumed that the fractional derivative's order is related to the turbulence intensity, from which a correlation between this index and the Reynolds number was obtained. The velocity profile obtained was used to predict the thickness behavior of the laminar sublayer. This parameter is related to the efficiency of mass and energy transport processes during industrial equipment operations in chemical engineering.

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Conflict of interest

The authors declare no conflict of interest.

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